LINEABILITY AND ADDITIVITY OF ALMOST INJECTIVE FUNCTIONS

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Abstract. We discuss additivity and lineability of classes of functions generalizing the concept of injectivity. In particular, we prove that it is consistent with ZFC that the class of almost injective functions has maximal possible lineability, i.e., it contains (with the exception of the zero function) a vector space of cardinality $2^{\mathfrak{c}}$.

1. Introduction

As usual, the symbols $\mathbb{N}$ and $\mathbb{R}$ will denote the sets of positive integers and real numbers, respectively. The cardinality of a set X is denoted by the symbol $|X|$. In particular, $|\mathbb{N}|$ is denoted by ω and $|\mathbb{R}|$ is denoted by $\mathfrak{c}$. We consider only real-valued functions. No distinction is made between a function and its graph. We write $f|A$ for the restriction of f to the set $A \subseteq \mathbb{R}$. The symbol χ_A denotes the characteristic function of the set A. For any subset Y of a vector space V and any $v \in V$ we define $v + Y = \{v + y \colon y \in Y\}$.

We will recall now some definitions related to lineability (see [1,2,4,8]). Let V be a vector space over $\mathbb{R}$, $\mathcal{F} \subseteq V$, and κ be a cardinal number. We say $\mathcal{F}$ is *κ-lineable* if $\mathcal{F} \cup \{0\}$ contains a subspace of V of dimension κ. The *(coefficient of) lineability* of the subset $\mathcal{F}$ is denoted by $\mathcal{L}(\mathcal{F})$ and defined as follows

$$\mathcal{L}(\mathcal{F}) = \min\{\kappa \colon \mathcal{F} \text{ is not } \kappa\text{-lineable}\}.$$

The additivity $\mathrm{A}(\mathcal{F})$ of $\mathcal{F} \subsetneq \mathbb{R}^X$ is defined as the smallest cardinality of a family $\mathcal{G} \subseteq \mathbb{R}^X$ for which there is no $g \in \mathbb{R}^X$ such that $g + \mathcal{G} \subseteq \mathcal{F}$ (see [7]). It was investigated for many classes of real functions.

The concept of a one-to-one (injective) function is a very fundamental one and is used in most branches of mathematics. Many results dealing with various types of transformations require the assumption of injectivity. One may wonder to which extent this assumption

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can be weakened in certain situations. In [5, 6], the author investigates certain topological properties that are being preserved by continuous *almost injective functions* (functions for which the set of points whose preimage has more than one point is countable). Motivated by this idea, we will study additivity and lineability of classes of functions related to almost injectivity. Specifically, we will consider the following families of real functions (X is a set and κ is a cardinal number such that $\kappa \leq |X|$).

$$\mathrm{SAI}(X) = \{f\colon X \to \mathbb{R}\colon \exists_{A\subseteq X}\ \ |A| < \omega \text{ and } f|(X \setminus A) \text{ is injective}\}$$
$$\mathrm{AI}(X) = \{f\colon X \to \mathbb{R}\colon \exists_{A\subseteq X}\ \ |A| \leq \omega \text{ and } f|(X \setminus A) \text{ is injective}\}$$
$$\mathrm{WAI}(X) = \{f\colon X \to \mathbb{R}\colon \exists_{A\subseteq X}\ \ |A| < \mathfrak{c} \text{ and } f|(X \setminus A) \text{ is injective}\}$$
$$F_{<\kappa}(X) = \{f\colon X \to \mathbb{R}\colon \forall_{y\in\mathbb{R}}\ \ |f^{-1}(y)| < \kappa\}$$
$$F_{\kappa}(X) = \{f\colon X \to \mathbb{R}\colon \forall_{y\in\mathbb{R}}\ \ |f^{-1}(y)| = 0 \text{ or } \kappa\}$$
$$F_{\leq\kappa}(X) = \{f\colon X \to \mathbb{R}\colon \forall_{y\in\mathbb{R}}\ \ |f^{-1}(y)| \leq \kappa\}$$

If $X = \mathbb{R}$, we will use the symbols $\mathrm{SAI}, \mathrm{AI}$, $\mathrm{WAI}, F_{<\kappa}, F_{\kappa}$, and $F_{\leq\kappa}$, respectively. Note that $F_1(X)$ is a family of injective functions on X. We will refer to functions in $\mathrm{SAI}(X)$, $\mathrm{AI}(X)$, and $\mathrm{WAI}(X)$ as strongly almost injective, almost injective, and weakly almost injective, respectively. Obviously $F_\omega(X) \subseteq F_{\leq\omega}(X) \subseteq F_{<\mathfrak{c}}$, $\mathrm{AI}(X) \subseteq F_{\leq\omega}(X)$, and $\mathrm{SAI}(X) \subseteq \mathrm{AI}(X) \subseteq \mathrm{WAI}(X) \subseteq F_{<\mathfrak{c}}$.

The conditions used in the definitions of the above families can be thought of as generalizations of the concept of injectivity. The problem of additivity and lineability of injective functions is quite trivial as it can be easily established that $\mathrm{A}(F_1(X)) = \mathfrak{c}$ and $\mathcal{L}(F_1(X)) = 2$ for any set X with at least two elements (see [10]). We will present some results on additivity and lineability of functions representing generalized injectivity (the additivity and lineability of $F_n, F_{\leq n}$ $(n \in \omega)$, $F_{<\omega}$ were studied in [10] and $F_{<\mathfrak{c}}$ in [4]).

2. Additivity

We start by investigating $\mathrm{A}(F_\omega)$.

Fact 2.1. $\mathrm{A}(F_\omega) \leq \mathfrak{c}$.

Proof. Let $\mathcal{F} = \{f \in \mathbb{R}^{\mathbb{R}}\colon f|(\mathbb{R} \setminus X) \equiv 0, X \subseteq \mathbb{R}, |X| = \omega\}$. Note that $|\mathcal{F}| = \mathfrak{c}$ and $f \in \mathcal{F}$ for $f \equiv 0$. Now pick any $g \in \mathbb{R}^{\mathbb{R}}$ and assume that $g \notin F_\omega$. Then $g + \mathcal{F} \not\subseteq F_\omega$. Next, if $g \in F_\omega$, pick $y_0 \in g[\mathbb{R}]$ and $x_0 \in g^{-1}(y_0)$. Define $f = \chi_{(g^{-1}(y_0)\setminus\{x_0\})}$ and observe that $(g+f)^{-1}(y_0) = \{x_0\}$. Therefore $g + f \notin F_\omega$ and consequently $g + \mathcal{F} \not\subseteq F_\omega$ as $f \in \mathcal{F}$. □

Next we will work on finding a lower bound for $\mathrm{A}(F_\omega)$ by utilizing the following two lemmas. The first of these lemmas uses the assumption of regularity of $\mathfrak{c}$: cof($\mathfrak{c}$)=$\mathfrak{c}$ (for more detail see [3]).

Lemma 2.2. (cof($\mathfrak{c}$)=$\mathfrak{c}$) *Let Y be a set of cardinality of $\mathfrak{c}$ and $\{f_i\}_1^n \subseteq \mathbb{R}^Y$, $n = 1, 2, \ldots$. Then there exists an $X \subseteq Y$ such that $|X| = \mathfrak{c}$ and for every $i \leq n$, $f_i|X$ is injective or constant.*

Proof. This lemma easily follows from the observation that, under the assumption of cof($\mathfrak{c}$)=$\mathfrak{c}$, every function from $\mathbb{R}^Y$ is constant or injective on a set of cardinality $\mathfrak{c}$. □

Example 2.3. (CH) *There exists an infinite family $\{f_n\}_{n<\omega} \subseteq \mathbb{R}^{\mathbb{R}}$ for which the conclusion of Lemma 2.2 fails.*

Proof. The Continuum Hypothesis implies the existence of an Ulam matrix on $\mathbb{R}$ (see [9]), i.e., a family $\{M^n_\xi : n < \omega, \xi < \mathfrak{c}\}$ of subsets of $\mathbb{R}$ with

$$M^n_\xi \cap M^n_\alpha = \emptyset, \text{ for } n < \omega,\ \xi < \alpha < \mathfrak{c},$$

$$\text{the complement of } \bigcup_{n<\omega} M^n_\xi \text{ is a countable set, for } \xi < \mathfrak{c}.$$

Fix an enumeration $\{x_\xi : \xi < \mathfrak{c}\}$ of $\mathbb{R}$. Define f_n as an extension of $\bigcup_{\xi<\mathfrak{c}} x_\xi \chi_{M^n_\xi}$ onto $\mathbb{R}$, for every $n < \omega$. We are now in a position to show that $F = \{f_n \colon n < \omega\}$ is a counterexample for the conclusion of Lemma 2.2. Let $X \subseteq \mathbb{R}$ be a set of cardinality $\mathfrak{c}$. Since the complement of $\bigcup_{n<\omega} M^n_\xi$ is a countable set, we have that $|X \cap \bigcup_{n<\omega} M^n_\xi| = \mathfrak{c}$ (for $\xi < \mathfrak{c}$). Hence, for every $\xi < \mathfrak{c}$ there exists an $n < \omega$ such that $|X \cap M^n_\xi| = \mathfrak{c}$. Consequently, there exists an $n_0 < \omega$ such that $|X \cap M^{n_0}_\xi| = \mathfrak{c}$ for $\mathfrak{c}$-many ξ. Therefore, f_{n_0} is neither injective nor constant on X. □

Lemma 2.4. *Let $|X| = \mathfrak{c}$ and $\{f_i\}_1^n \subseteq \mathbb{R}^X$, $n = 1, 2, \ldots$ be such that f_1 is constant and f_i is injective for $i = 2, \ldots, n$. There exists a $g \in \mathbb{R}^X$ such that $g + f_1$ is bijective and $g + f_i$ is injective for $i = 2, \ldots, n$.*

Proof. The construction of g is through transfinite induction. First note that we can assume that $f_1 \equiv 0$. Let $X = \{x_\xi \colon \xi < \mathfrak{c}\}$ and $\mathbb{R} = \{y_\xi \colon \xi < \mathfrak{c}\}$ and assume that the construction of g is carried out for all $\alpha < \beta < \mathfrak{c}$ in such a way that $|\mathrm{dom}(g)| \le \max\{\omega, \beta\}$, $\{x_\xi \colon \xi < \beta\} \subseteq \mathrm{dom}(g)$, $\{y_\xi \colon \xi < \beta\} \subseteq \mathrm{range}(g)$, and $g + f_1, \dots, g + f_n$ are injective. First, if $x_\beta \notin \mathrm{dom}(g)$, then choose $g(x_\beta) \notin \{g(x) + f_i(x) - f_i(x_\beta) \colon x \in \mathrm{dom}(g) \text{ and } i = 1, \dots, n\}$. Next, if $y_\beta \notin$ range(g), then choose

$$x' \notin \mathrm{dom}(g) \cup \bigcup_{i=2}^{i=n} f_i^{-1}(\{g(x) + f_i(x) - y_\beta \colon x \in \mathrm{dom}(g)\})$$

and define $g(x') = g(x') + f_1(x') = y_\beta$. It is easy to see that the function g defined through the above construction has the required properties. □

Before stating the next result, let us recall that a function $f \colon \mathbb{R} \to \mathbb{R}$ is called *everywhere surjective* ($f \in \mathrm{ES}$) if $f(I) = \mathbb{R}$ for every non-trivial open interval I.

Theorem 2.5. *Assume that* $\mathrm{cof}(\mathfrak{c}) = \mathfrak{c}$. *Let* Y *be a set of cardinality* $\mathfrak{c}$ *and* $\{f_i\}_1^n \subseteq \mathbb{R}^Y$, $n \in \{1, 2, \dots\}$. *There exists a* $g \in \mathbb{R}^Y$ *such that* $g + f_1$ *is surjective and* $g + f_i \in F_{\le\omega}(Y)$ *for* $i = 1, 2, \dots, n$. *In particular,* $\mathrm{A}(F_\omega) \ge \omega$ *and* $\mathrm{A}(F_\omega \cap \mathrm{ES}) \ge \omega$.

Proof. By Lemma 2.2, there exists an $X \subsetneq Y$ such that $|X| = \mathfrak{c}$ and $(f_i - f_1)|X$ is either injective or constant for $i = 1, 2, \dots, n$. Next, by applying Lemma 2.4, we can conclude that there is a $g' \colon X \to \mathbb{R}$ such that $g' + (f_1 - f_1)$ is bijective and $g' + (f_i - f_1)$ is injective for $i = 2, \dots, n$. Now, since $\mathrm{A}(F_1(Y \setminus X)) = \mathfrak{c}$, there exists a $g'' \colon (Y \setminus X) \to \mathbb{R}$ such that $g'' + (f_i - f_1)$ is injective for $i = 1, \dots, n$. Define $g = (g' - f_1) \cup (g'' - f_1)$ and notice that $g + f_1$ is surjective and $g + f_i \in F_{\le\omega}(Y)$ for $i = 1, 2, \dots, n$.

To see $\mathrm{A}(F_\omega) \ge \omega$, choose an arbitrary $\{f_i\}_1^n \subseteq \mathbb{R}^\mathbb{R}$ and decompose $\mathbb{R} = \bigcup_{k=1}^\infty (Y_k^1 \cup \dots \cup Y_k^n)$ into sets of cardinality $\mathfrak{c}$. Then apply the main conclusion of this theorem to each Y_k^i to obtain a $g_k^i \colon Y_k^i \to \mathbb{R}$ such that $g_k^i + f_i$ is surjective and $g_k^i + f_i \in F_{\le\omega}(Y_k^i)$ for $k \le \omega$ and $i = 1, 2, \dots, n$. It is easy to observe that by defining $g = \bigcup_{k=1}^\infty (g_k^1 \cup \dots \cup g_k^n)$ we obtain a function $g \colon \mathbb{R} \to \mathbb{R}$ such that $g + f_i \in F_\omega$ for $i = 1, 2, \dots, n$. Observe that, by appropriate choice of the sets Y_k^i it can be assured that all the functions $g + f_i$ are everywhere surjective.
□

Theorem 2.6.

(i) *If* $2^{\omega_1} = \mathfrak{c}$ *(e.g., under MA and* $\neg$*CH), then* $\mathrm{A}(\mathrm{AI}) = \mathrm{A}(F_{\leq\omega}) = \mathfrak{c} = \mathrm{A}(F_1)$.

(ii) $\mathrm{A}(\mathrm{WAI}) > \mathrm{A}(\mathrm{SAI}) = \mathfrak{c}$.

(iii) *If CH holds, then* $\mathrm{A}(F_{\leq\omega}) \geq \mathrm{A}(\mathrm{AI}) = \mathrm{A}(\mathrm{WAI}) > \mathfrak{c}$. *Furthemore, under GCH we have that* $\mathrm{A}(F_{\leq\omega}) = \mathrm{A}(\mathrm{AI}) = \mathrm{A}(\mathrm{WAI}) = \mathfrak{c}^+ = 2^{\mathfrak{c}}$.

Proof. (i) First, notice that $\mathfrak{c} = \mathrm{A}(F_1) \leq \mathrm{A}(\mathrm{AI}) \leq \mathrm{A}(F_{\leq\omega})$ since $F_1 \subseteq \mathrm{AI} \subseteq F_{\leq\omega}$. Next we will justify that $\mathrm{A}(F_{\leq\omega}) \leq \mathfrak{c}$. Fix $X \subseteq \mathbb{R}$ such that $|X| = \omega_1$ and define $\mathcal{F} = \{f \in \mathbb{R}^{\mathbb{R}} \colon f|(\mathbb{R}\setminus X) \equiv 0\}$. It is easy to see that for any $g \in \mathbb{R}^{\mathbb{R}}$, $g + \mathcal{F} \not\subseteq F_{\leq\omega}$. Since we assumed $2^{\omega_1} = \mathfrak{c}$ we also have that $|\mathcal{F}| = \mathfrak{c}$.

(ii) Choose an arbitrary family of real functions $\mathcal{G} = \{g_\beta \colon \beta < \mathfrak{c}\}$. Using the transfinite induction, it is not difficult to construct a function $g \in \mathbb{R}^{\mathbb{R}}$ such that $g + \mathcal{G} \subseteq \mathrm{WAI}$. Indeed, let $\mathbb{R} = \{x_\xi \colon \xi < \mathfrak{c}\}$, define $g(x_0)$ arbitrarily, and assume that g is defined on $\{x_\xi \colon \xi < \alpha\}$ for some $\alpha < \mathfrak{c}$. We choose $g(x_\alpha) \notin \bigcup_{\beta\leq\alpha} -g_\beta(x_\alpha) + \{g(x_\xi) + g_\beta(x_\xi) \colon \xi < \alpha\}$. The above choice assures that $(g + g_\alpha)|\{x_\xi \colon \xi \geq \alpha\}$ is injective.

To see $\mathrm{A}(\mathrm{SAI}) = \mathfrak{c}$ note that $F_1 \subseteq \mathrm{SAI} \subseteq F_{<\omega}$ and recall that $\mathrm{A}(F_1) = \mathrm{A}(F_{<\omega}) = \mathfrak{c}$ (see [10]).

(iii) This part easily follows from part (ii) and the observation that, under CH, we obviously have $\mathrm{AI} = \mathrm{WAI}$. □

Observe that the above results imply that it cannot be proved in ZFC that $\mathrm{A}(F_1)$ is equal to $\mathrm{A}(\mathrm{AI})$ and neither it can be proved that the two additivities are different. Note also that it cannot be proved in ZFC that $\mathrm{A}(F_\omega)$ is equal to $\mathrm{A}(F_{\leq\omega})$. However, it remains an open problem whether it is consistent with ZFC that $\mathrm{A}(F_\omega) = \mathrm{A}(F_{\leq\omega})$. Additionally, Theorem 2.6 implies that $\mathrm{A}(\mathrm{AI}) = \mathrm{A}(\mathrm{WAI})$ is independent of ZFC.

3. Lineability

In this section we turn our attention to lineability. Before stating the first result of this section, let us recall some additional definitions. A function $f \colon \mathbb{R} \to \mathbb{R}$ is called *strongly everywhere surjective* ($f \in \mathrm{SES}$) if $|f^{-1}(y) \cap I| = \mathfrak{c}$ for every $y \in \mathbb{R}$ and non-trivial open interval I. A function $g \colon \mathbb{R} \to \mathbb{R}$ is called *additive* ($g \in \mathrm{AD}$) if it is linear over the field of rational numbers.

Remark 3.1. $\mathcal{L}(F_\omega) = \mathcal{L}(F_{\leq\omega}) \geq \mathcal{L}(F_{<\omega}) = \mathfrak{c}^+$. Additionally, if CH holds, then $\mathrm{AD} \cap \mathrm{ES} \setminus \mathrm{SES} \subseteq F_\omega$ and, as a consequence, $\mathcal{L}(F_\omega) = \mathcal{L}(F_{\leq\omega}) > \mathfrak{c}^+$.

Proof. Obviously $\mathcal{L}(F_\omega) \leq \mathcal{L}(F_{\leq\omega})$. Now, let $\kappa < \mathcal{L}(F_{\leq\omega})$, $\{X_\xi \colon \xi < \mathfrak{c}\}$ be a decomposition of $\mathbb{R}$ into denumerable sets, and $X = \{x_\xi \colon \xi < \mathfrak{c}\}$ be such that $x_\xi \in X_\xi$ for $\xi < \mathfrak{c}$. There exists a vector space V of cardinality κ such that $V \subseteq F_{\leq\omega}(X) \cup \{0\}$. Define $W = \{f \in \mathbb{R}^{\mathbb{R}} \colon \exists_{g\in V}\ \forall_{\xi<\mathfrak{c}}\ \ f|X_\xi \equiv g(x_\xi)\}$. It is easy to check that $W \subseteq F_\omega \cup \{0\}$.

The inequality $\mathcal{L}(F_{\leq\omega}) \geq \mathcal{L}(F_{<\omega})$ is obvious and the equality $\mathcal{L}(F_{<\omega}) = \mathfrak{c}^+$ was proved in [10] (Theorem 2.3) .

To see that, under the assumption of CH we have $\mathrm{AD} \cap \mathrm{ES} \setminus \mathrm{SES} \subseteq F_\omega$, note that for any additive function f, $|f^{(-1)}(y)| = |\ker(f)|$ for all $y \in$ range(f). Therefore, if $f \in \mathrm{AD} \cap \mathrm{ES} \setminus \mathrm{SES}$, then $|f^{(-1)}(y)| = |\ker(f)| = \omega$ for all $y \in \mathbb{R}$, which implies that $f \in F_\omega$. Finally, it was proved in [11] that CH implies $\mathcal{L}(\mathrm{AD} \cap \mathrm{ES} \setminus \mathrm{SES}) > \mathfrak{c}^+$. □

Theorem 3.2. $\mathcal{L}(\mathrm{WAI}) \geq \mathcal{L}(\mathrm{AD} \cap \mathrm{ES} \setminus \mathrm{SES})$.

Proof. Fix a Hamel basis $H \subseteq \mathbb{R}$. Observe that if $f \in \mathrm{AD} \cap \mathrm{ES} \setminus \mathrm{SES}$, then $f|H \in \mathrm{WAI}(H)$. Indeed, if $f|H \notin \mathrm{WAI}(H)$, then $\{x \in H \colon |(f|H)^{-1}(f(x))| > 1\}| = \mathfrak{c}$. This implies that $|\ker(f)| = \mathfrak{c}$ and consequently $f^{-1}(y)$ is a $\mathfrak{c}$-dense set for every $y \in \mathbb{R}$. This would contradict the fact that $f \notin \mathrm{SES}$.

Now, let $\kappa < \mathcal{L}(\mathrm{AD} \cap \mathrm{ES} \setminus \mathrm{SES})$ and $W \subseteq \mathrm{AD} \cap (\mathrm{ES} \setminus \mathrm{SES}) \cup \{0\}$ be a vector space of cardinality κ. Next, fix a bijection $b \colon \mathbb{R} \to H$ and define

$$V = \{(h|H) \circ b \colon h \in W\}.$$

It is easy to check that $|V| = |W| = \kappa$ and $V \subseteq \mathrm{WAI} \cup \{0\}$. Consequently,

$$\mathcal{L}(\mathrm{WAI}) \geq \mathcal{L}(\mathrm{AD} \cap \mathrm{ES} \setminus \mathrm{SES}).$$

□

Using the above theorem and [11, Theorem 2] we obtain the following corollary.

Corollary 3.3. *If $\mathfrak{c}$ is regular, then $\mathcal{L}(\mathrm{WAI}) > \mathfrak{c}^+$. Hence, it is consistent with ZFC (e.g., under the assumption of GCH) that $\mathcal{L}(\mathrm{WAI}) = \mathcal{L}(\mathrm{AI}) = (2^{\mathfrak{c}})^+$.*

Before investigating $\mathcal{L}(\mathrm{SAI})$, we introduce some notation and prove auxiliary results. Let $X = \{x_\alpha\colon \alpha < \kappa\}$ for some cardinal κ and $\mathcal{F}$ be a family of real valued functions on X. For $x, y \in X$ and $f_1, \dots, f_n \in \mathcal{F}$ we define $v^{x,y}_{f_1,\dots,f_n} = (f_1(y), \dots, f_n(y)) - (f_1(x), \dots, f_n(x))$. Additionally, for $Y \subseteq X$ we set

$$E^Y_{f_1\dots,f_n} = \{v^{x_{\alpha_1},x_{\alpha_2}}_{f_1,\dots,f_n}\colon \alpha_1 < \alpha_2, x_{\alpha_1}, x_{\alpha_2} \in Y\}.$$

Furthermore, we will say that $\mathcal{F}$ *satisfies condition* (I) *on* Y if

(I) for any distinct $f_1, \dots, f_n \in \mathcal{F}$ and any distinct $(x_{\alpha^1_1}, x_{\alpha^1_2}), \dots, (x_{\alpha^n_1}, x_{\alpha^n_2}) \in Y^2$ (where $\alpha^i_1 < \alpha^i_2$, $i \le n \ge 1$) we have $\dim(\mathrm{span}(\{v^{x_{\alpha^i_1},x_{\alpha^i_2}}_{f_1,\dots,f_n}\colon i \le n\})) = n$.

Lemma 3.4. *Let $\mathcal{F}$ be a family of functions on X. $\mathcal{F}$ satisfies the condition* (I) *on X if and only if for any $a_1, \dots, a_n \in \mathbb{R}$ (not all equal to 0) and any distinct $f_1, \dots, f_n \in \mathcal{F}$ $(n \ge 1)$ there exists a set $A \subseteq X$ such that $|A| \le n-1$ and $(a_1 f_{\xi_1} + \dots + a_n f_{\xi_n})|(\mathbb{R} \setminus A)$ is injective.*

Proof. Assume that for some $n \ge 1$, some $a_1, \dots, a_n \in \mathbb{R}$ (not all equal to 0), and some distinct $f_1, \dots, f_n \in \mathcal{F}$ there is no set $A \subseteq X$ such that $|A| \le n-1$ and $(a_1 f_{\xi_1} + \dots + a_n f_{\xi_n})|(\mathbb{R} \setminus A)$ is injective. This is equivalent to the existence of n distinct pairs $(x_{\alpha^i_1}, x_{\alpha^i_2})$ ($\alpha^i_1 < \alpha^i_2$ for $i = 1, \dots, n$) for which

$$a_1 f_1(x_{\alpha^i_2}) + \dots + a_n f_n(x_{\alpha^i_2}) = a_1 f_1(x_{\alpha^i_1}) + \dots + a_n f_n(x_{\alpha^i_1}) \ (i \le n).$$

Consequently, $\dim(\mathrm{span}(\{v^{x_{\alpha^i_1},x_{\alpha^i_2}}_{f_1,\dots,f_n}\colon i \le n\}) < n$ as

$$\mathrm{span}(\{v^{x_{\alpha^i_1},x_{\alpha^i_2}}_{f_1,\dots,f_n}\colon i \le n\}) \subseteq \{(t_1, \dots, t_n) \in \mathbb{R}^n\colon a_1 t_1 + \dots + a_n t_n = 0\}.$$

Equivalently, we obtain the negation of the condition (I). □

Theorem 3.5. *There exists a family of real functions $\mathcal{F}$ of cardinality $\mathfrak{c}$ that satisfies the condition* (I) *on $\mathbb{R}$.*

Before proving the above theorem we state and justify the following corollary.

Corollary 3.6. $\mathcal{L}(\mathrm{WAI}) \ge \mathcal{L}(\mathrm{AI}) \ge \mathcal{L}(\mathrm{SAI}) = \mathfrak{c}^+$.

Proof. The inequalities are obvious. To see $\mathcal{L}(\mathrm{SAI}) \ge \mathfrak{c}^+$ note that for the family $\mathcal{F}$ from Theorem 3.5 we have, by Lemma 3.4, that $\mathrm{span}(\mathcal{F}) \subseteq \mathrm{SAI} \cup \{0\}$ and $\dim(\mathrm{span}(\mathcal{F})) = \mathfrak{c}$. The inequality $\mathcal{L}(\mathrm{SAI}) \le \mathfrak{c}^+$ is a consequence of $\mathrm{SAI} \subseteq F_{<\omega}$ and $\mathcal{L}(F_{<\omega}) = \mathfrak{c}^+$ (see [10]). □

Proof of Theorem 3.5. Let $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ ($|\mathcal{F}| < \mathfrak{c}$) be a family of functions satisfying the condition (I) on $\mathbb{R}$. We will define a real function $g \notin \mathcal{F}$ such that $\mathcal{F} \cup \{g\}$ also satisfies the condition (I) on $\mathbb{R}$.

Fix $\mathbb{R} = \{x_\alpha \colon \alpha < \mathfrak{c}\}$ and choose $g(x_0) \notin \{f(x_0) \colon f \in \mathcal{F}\}$. Now let $\beta < \mathfrak{c}$ and assume that g is defined on $X = \{x_\alpha \colon \alpha < \beta\}$ in such a way that $\{f|X \colon f \in \mathcal{F}\} \cup \{g\}$ satisfies the condition (I) on X.

Next we will define $g(x_\beta)$. For $n \geq 1$, distinct $f_1 \ldots, f_n \in \mathcal{F}$, $x \in X = \{x_\alpha \colon \alpha < \beta\}$, and $\{v_1, \ldots, v_n\} \subseteq E^X_{f_1,\ldots,f_n,g}$, let $t^x_{v_1,\ldots,v_n}$ denote a real number such that the only element of

$$\{(f_1(x_\beta), \ldots, f_n(x_\beta), t) \in \mathbb{R}^{n+1} \colon t \in \mathbb{R}\} \cap ((f_1(x), \ldots, f_n(x), g(x)) + \operatorname{span}\{v_1, \ldots, v_n\})$$

is $(f_1(x_\beta), \ldots, f_n(x_\beta), t^x_{v_1,\ldots,v_n})$. Now choose

$$g(x_\beta) \in \mathbb{R} \setminus \bigcup_{n\geq 1} \bigcup_{\substack{f_1,\ldots,f_n \in \mathcal{F} \\ \text{distinct}}} \bigcup_{\{v_1,\ldots,v_n\} \subseteq E^X_{f_1,\ldots,f_n,g}} \bigcup_{\alpha<\beta} \{g(x_\alpha), t^{x_\alpha}_{v_1,\ldots,v_n}\}.$$

To see that $\{f|X \cup \{x_\beta\} \colon f \in \mathcal{F}\} \cup \{g\}$ satisfies the condition (I) on $X \cup \{x_\beta\}$ assume that for some $n \geq 1$, distinct $f_1, \ldots, f_n \in \mathcal{F}$, and distinct $(x_{\alpha^1_1}, x_{\alpha^1_2}), \ldots, (x_{\alpha^{n+1}_1}, x_{\alpha^{n+1}_2})$ (where $\alpha^i_1 < \alpha^i_2 \leq \beta$, $i \leq n+1$) we have $\dim(\operatorname{span}(\{v^{x_{\alpha^i_1}, x_{\alpha^i_2}}_{f_1,\ldots,f_n,g} \colon i \leq n+1\})) < n+1$. Note that $\beta = \max\{\alpha^i_k \colon k = 1, 2 \text{ and } i = 1, \ldots, n+1\}$ as by the inductive assumption $\{f|X \colon f \in \mathcal{F}\} \cup \{g|X\}$ satisfies the condition (I) on X. We can assume that β is an element of exactly one of the pairs $(\alpha^1_1, \alpha^1_2), \ldots, (\alpha^{n+1}_1, \alpha^{n+1}_2)$. Indeed, if $\beta = \alpha^i_2 = \alpha^j_2$ for some $i, j \leq n+1$ with $\alpha^i_1 < \alpha^j_1$, then the pair $(\alpha^i_1, \alpha^i_2) = (\alpha^i_1, \beta)$ could be replaced by (α^i_1, α^j_1) as

$$\operatorname{span}(\{v^{x_{\alpha^i_1}, x_\beta}_{f_1,\ldots,f_n,g}, v^{x_{\alpha^j_1}, x_\beta}_{f_1,\ldots,f_n,g}\}) = \operatorname{span}(\{v^{x_{\alpha^i_1}, x_{\alpha^j_1}}_{f_1,\ldots,f_n,g}, v^{x_{\alpha^j_1}, x_\beta}_{f_1,\ldots,f_n,g}\}).$$

For convenience we can assume that $\alpha^{n+1}_2 = \beta$. Now, based on the inductive assumption, we conclude that $\dim(\operatorname{span}(\{v^{x_{\alpha^i_1}, x_{\alpha^i_2}}_{f_1,\ldots,f_n,g} \colon i \leq n\})) = n$ and, as a consequence, $v^{x_{\alpha^j_1}, x_\beta}_{f_1,\ldots,f_n,g} \in \operatorname{span}(\{v^{x_{\alpha^i_1}, x_{\alpha^i_2}}_{f_1,\ldots,f_n,g} \colon i \leq n\})$. This implies that

$$(f_1(x_\beta), \ldots, f_n(x_\beta), g(x_\beta)) \in (f_1(x^1_{\alpha_{n+1}}), \ldots, f_n(x^1_{\alpha_{n+1}}), g(x^1_{\alpha_{n+1}})) + \operatorname{span}(\{v^{x_{\alpha^i_1}, x_{\alpha^i_2}}_{f_1,\ldots,f_n,g} \colon i \leq n\})$$

which contradicts the way $g(x_\beta)$ was chosen.

This completes the inductive step and the definition of g. It is easy to see that $g \notin \mathcal{F}$ and $\mathcal{F} \cup \{g\}$ satisfies the condition (I) on $\mathbb{R}$. □

We finish by stating some open problems.

Problem 3.7. Is it consistent with ZFC that $\mathrm{A}(F_\omega) = \mathrm{A}(F_{\leq\omega})$?

Problem 3.8. Can it be proved in ZFC that $\mathcal{L}(\mathrm{WAI}) = (2^{\mathfrak{c}})^+$?

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